SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

November 2007

First Assessment

Mathematics
Extension

General Instructions
- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question

Total Marks – 60
- All Questions may be attempted
- Each Question is worth 15 marks, and should be handed up in a separate examination Booklet.
- Full Marks may not be awarded for careless or poorly set out work.

Examiner – *A.M.Gainford*

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Question 1. (15 Marks) (**Start a new booklet**.)

Marks

(a) (i) In how many ways may all the letters of the word ANAGRAM be arranged in a line? **2**

(ii) In how many ways may four men and four women be arranged around a circular table, if no two men are to be seated next to each other?

(b) The point $P(8, -16)$ divides the interval AB externally in the ratio 2:1. Find the coordinates of the point B given that A is $(-2, 6)$. **3**

(c) Solve the following inequalities, and in each case graph the solution on the number line: **6**

(i) $\frac{2}{x} > \frac{1}{3}$

(ii) $\frac{x}{x-1} \leq 2$

(d) Consider the equation $3\sin\theta + 4\cos\theta = 2$. **4**

(i) Express the equation in the form $R\sin\left(\theta+\alpha\right) = 2$, where R is a constant, and α is constant and acute.

(ii) Hence or otherwise solve the equation for $0^\circ \leq \theta \leq 360^\circ$, correct to the nearest minute.

Question 2. (15 Marks) (**Start a new booklet**.)

Marks

(a) Solve the following equation for $0^\circ \le \theta \le 360^\circ$: **2**

$$6\cos^2\theta = 4 + \sin\theta\ .$$

Give your answers correct to the nearest minute.

(b) Given the polynomial $P(x) = 2x^3 - 6x^2 - 12x + 16$: **4**

(i) Use the factor theorem to determine a zero of the polynomial.

(ii) Express $P(x)$ as a product of linear factors.

(c) Show that: **2**

$$2\sec\theta\left(\sec\theta - \tan\theta\right) - 1 = \frac{1-\sin\theta}{1+\sin\theta}$$

(d) Find the acute angle between the lines $x + 2y - 6 = 0$ and $3x - 2y + 7 = 0$, correct to the nearest minute. **3**

(e) Write in polynomial form the equation of the monic quartic with a double root at $x = 2$, and simple roots at $x = 0$ and $x = -3$. **2**

(f) When a certain polynomial is divided by $(x+2)(x-3)$ the remainder is $2x+3$. What are the remainders when it is divided by $x+2$ and $x-3$ separately? **2**

Question 3. (15 Marks) (**Start a new booklet**.)

Marks

(a) Solve the equation $x^4 + 4x^3 - 16x - 16 = 0$ by factorization, or otherwise. **3**

(b) From a point *A* on the ground due South of the base *G* of a television tower, the angle of elevation of the top (*T*) of the tower is $16^\circ 42'$. One kilometre East of *A*, at the point *B*, the angle of elevation of the top of the tower is 12°. **4**

(i) Draw a neat diagram representing this situation.

(ii) Find the height of the tower to the nearest metre.

(c) *P* is a point on the parabola $x^2 = 4ay$ and *G* is the point where the normal at *P* intersects the axis of the parabola. If the line through *P* perpendicular to the axis of the parabola meets the axis at *N*, then the interval *NG* is called the subnormal corresponding to *P*. **4**

(i) Show that the point $P(2ap, ap^2)$ lies on the parabola for all values of *p*.

(ii) Derive the equation of the normal at *P*.

(iii) Prove that the length of the subnormal corresponding to *P* is constant, and equal to twice the focal length.

(d) Solve the inequality $|2x-1| > |x+2|$. **2**

(e) Find the general solution of the equation $\cos\theta \sin\theta = \frac{1}{2}$. **2**

Question 4. (15 Marks) (**Start a new booklet**.)

Marks

(a) Five persons are to be seated in a row. **3**

(i) How many arrangements are possible?

(ii) What is the probability that person A and person B are separated by exactly one other person?

(b) P is a variable point on the parabola $x^2 = 4ay$. The tangent at P meets the x-axis at Q and the y-axis at R. **4**
Determine the Cartesian equation of the locus of M, the midpoint of QR as P moves on the parabola.

(c) The ropes of a swing are 4m long and at rest the seat is $0 \cdot 6$ m above the ground. A child uses the swing, and the highest point reached on one side is 2m, and on the other side is $2 \cdot 3$ m above the ground. **4**

(i) Calculate, to the nearest degree, the angle through which the seat swings.

(ii) Find, to the nearest centimetre the straight line distance between the two highest points.

(d) (i) Find an expression for $\tan(\alpha + \beta + \gamma)$ in terms of $\tan\alpha$, $\tan\beta$, $\tan\gamma$. **4**

(ii) Put $\alpha = \beta = \gamma = \dfrac{\pi}{12}$ in the expression derived above, and show that $\tan\dfrac{\pi}{12}$ is a root of the equation $t^3 - 3t^2 - 3t + 1 = 0$.

This is the end of the paper.

2007 Mathematics Extension 1 Assessment 1: **Solutions Question 1**

1. (a) (i) In how many ways can the letters of the word ANAGRAM be arranged in a line? $\boxed{1}$

Solution: $\dfrac{7!}{3!} = \dfrac{5040}{6},$
$= 840.$

(ii) In how many ways can four men and four women be arranged around a circular table, if no two men are to be seated next to each other? $\boxed{1}$

Solution: $3!4! = 144.$

(b) The point $P(8, -16)$ divides the interval AB externally in the ratio $2:1$. Find the coordinates of the point B given that A is $(-2, 6)$. $\boxed{3}$

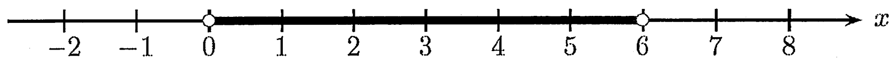
Solution:

$$8 = \frac{(-1)\times(-2)+2\times x}{-1+2}, \qquad -16 = \frac{(-1)\times 6 + 2\times y}{-1+2},$$
$$= 2+2x, \qquad = -6+2y,$$
$$\therefore\ x = 3. \qquad y = -5.$$
i.e. $B(3, -5)$.

(c) Solve the following inequalities, and in each case graph the solution on the number line:

(i) $\dfrac{2}{x} > \dfrac{1}{3}$ $\boxed{3}$

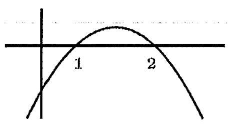
Solution: Clearly $x > 0$ (as $1/3 > 0$), $\therefore\ 6 > x$,
i.e. $0 < x < 6$.



(ii) $\dfrac{x}{x-1} \leq 2$ $\boxed{3}$

Solution: If $x-1 > 0,\ x > 1$; if $x-1 < 0,\ x < 1$;
then $x \leq 2x-2$, then $x \geq 2x-2$,
$x \geq 2$. $x \leq 2$.
$\therefore\ x \geq 2$. $\therefore\ x < 1$.

Alternative Solution: $x(x-1) \leq 2(x-1)^2,$
$(x-1)(x-2x+2) \leq 0,$
$(x-1)(2-x) \leq 0.$
$\therefore\ x < 1$ or $x \geq 2$.



Number line graph as above.

(d) Consider the equation $3\sin\theta + 4\cos\theta = 2$.

(i) Express the equation in the form $R\sin(\theta+\alpha) = 2$, where R is a constant, and α is constant and acute. $\boxed{2}$

Solution:
$$\begin{aligned} R &= \sqrt{3^2+4^2}, & \tan\alpha &= {}^4/_3, \\ &= 5. & \alpha &\approx 53.130102354°. \end{aligned}$$
$$i.e.\ 5\sin(\theta + \tan^{-1}({}^4/_3)) = 2.$$

(ii) Hence or otherwise solve the equation for $0° \le \theta \le 360°$, correct to the nearest minute. $\boxed{2}$

Solution:
$$\begin{aligned} \sin(\theta + 53°7'48.3685'') &\approx {}^2/_5, \\ \theta + 53°7'48.3685'' &\approx 23°34'41.4425'',\ 156°25'18.5575'', \\ & \quad 383°34'41.4425'',\ \ldots \\ \theta &\approx 103°17'30.1890'',\ 330°26'53.0740'', \\ &\approx 103°18',\ 330°27' \text{ (nearest minute)}. \end{aligned}$$

2) a)

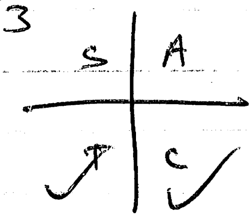
$$6\cos^2\theta = 4 + \sin\theta$$
$$6(1-\sin^2\theta) = 4+\sin\theta$$
$$6 - 6\sin^2\theta = 4 + \sin\theta$$
$$6\sin^2\theta + \sin\theta - 2 = 0$$

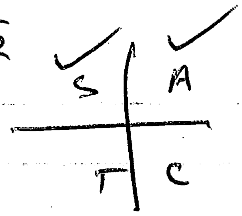
$\times$ -12
$+$ 1
$4, -3$

$$\frac{(6\sin\theta + 4)(6\sin\theta - 3)}{\underset{2\times 3}{6}} = 0$$

$$(3\sin\theta + 2)(2\sin\theta - 1) = 0$$

$\sin\theta = -\frac{2}{3}$ $\qquad\qquad$ $\sin\theta = \frac{1}{2}$





$\sin\alpha = \frac{2}{3}$ $\qquad\qquad$ $\sin\alpha = \frac{1}{2}$

$\alpha = 41°49'$ $\qquad\qquad$ $\alpha = 30°$

$$\theta = 30°, 150°, 221°49', 318°11'$$

b) i) $P(x) = 2x^3 - 6x^2 - 12x + 16$

$P(1) = 2(1)^3 - 6(1)^2 - 12(1) + 16$
$= 0$

$\therefore x = 1$ is a zero

ii) $\therefore x-1$ is a factor

$$
\begin{array}{r}
2x^2 - 4x - 16 \\
x-1 \,\big)\, 2x^3 - 6x^2 - 12x + 16 \\
\underline{2x^3 - 2x^2} \qquad\qquad\quad \\
-4x^2 - 12x \qquad\; \\
\underline{-4x^2 + 4x} \qquad\; \\
-16x + 16 \\
\underline{-16x + 16} \\
0
\end{array}
$$

$\therefore P(x) = (x-1)(2x^2 - 4x - 16)$

$P(x) = 2(x-1)(x^2 - 2x - 8)$

$\times$ -8
$+$ -2
$-4, 2$

$P(x) = 2(x-1)(x+2)(x-4)$

c) $LHS = 2\sec\theta(\sec\theta - \tan\theta) - 1$

$= \frac{2}{\cos\theta}\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right) - 1$

$= \frac{2(1-\sin\theta)}{\cos^2\theta} - 1$

$= \frac{2(1-\sin\theta)}{1-\sin^2\theta} - 1$

$= \frac{2(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} - 1$

$= \frac{2}{1+\sin\theta} - 1$

$= \frac{2-(1+\sin\theta)}{1+\sin\theta}$

$= \frac{1-\sin\theta}{1+\sin\theta}$

$= RHS$

$\therefore\ 2\sec\theta(\sec\theta - \tan\theta) - 1 = \frac{1-\sin\theta}{1+\sin\theta}$

d) $x + 2y - 6 = 0$

$2y = -x + 6$

$y = -\frac{x}{2} + 3$

$m_1 = -\frac{1}{2}$

$3x - 2y + 7 = 0$

$2y = 3x + 7$

$y = \frac{3}{2}x + \frac{7}{2}$

$m_2 = \frac{3}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\tan\theta = \left|\frac{\left(-\frac{1}{2}\right)-\left(\frac{3}{2}\right)}{1+\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)}\right|$$

$$\tan\theta = |-8|$$

$$\tan\theta = 8$$

$$\theta = 82^\circ 52'$$

e) $(x-2)^2 x (x+3) = 0$

$(x^2-4x+4)(x^2+3x) = 0$

$x^4-4x^3+4x^2+3x^3-12x^2+12x = 0$

$x^4-x^3-8x^2+12x = 0$

f) $P(x) = (x+2)(x-3)Q(x) + 2x+3$

$P(-2) = 2(-2)+3$

$= -1$

$P(3) = 2(3)+3$

$= 9$

$\therefore$ the remainders when $P(x)$ is divided by $x+2$ and $x-3$ are -1 and 9 respectively.

QUESTION 3. (EXT 1 YR 11 2007)

(a) $x^4 + 4x^3 - 16x - 16 = 0$

$\therefore x^4 - 16 + 4x^3 - 16x = 0$

$(x^2-4)(x^2+4) + 4x(x^2-4) = 0$

$(x^2-4)(x^2+4x+4) = 0$

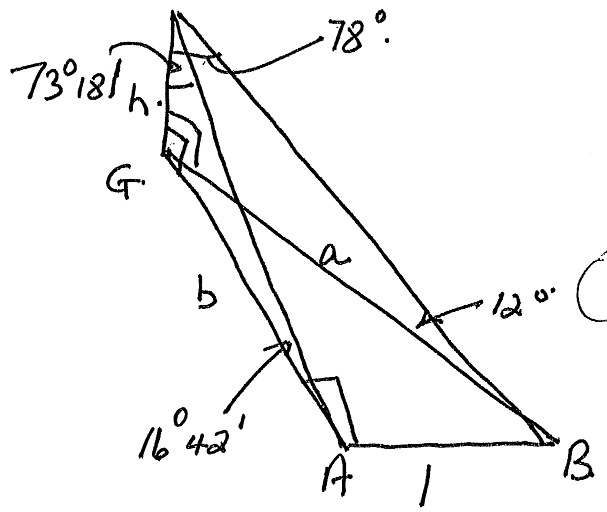
$(x-2)(x+2)(x+2)^2 = 0$

$(x-2)(x+2)^3 = 0$

$\boxed{x = 2, -2, -2, -2.}$

3

(b)



1

$b = h \tan 73°18'$

& $a = h \tan 78°$

$a^2 - b^2 = 1.$

$\therefore h^2\left[\tan^2 78° - \tan^2 73°18'\right] = 1$

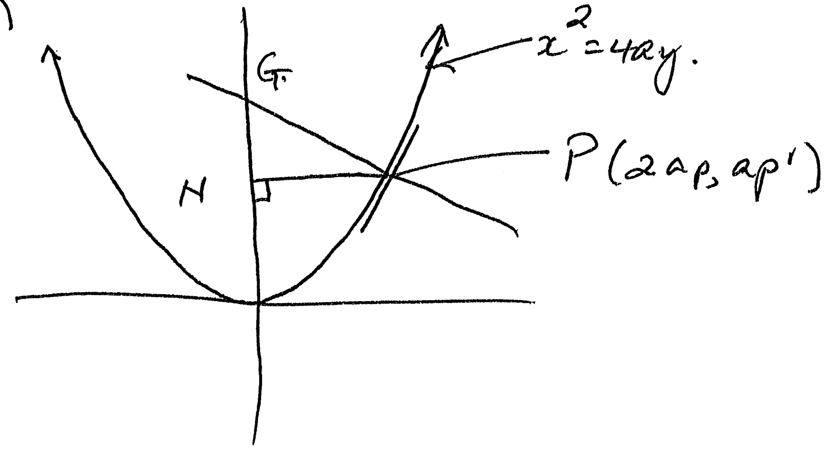
$h^2 = \dfrac{1}{\tan^2 78° - \tan^2 73°18'}$

$h = \dfrac{1}{\sqrt{\tan^2 78° - \tan^2 73°18'}}$

$\boxed{h \doteqdot 301\,m.}$

3

(c)



(i) $x^2=(2ap)^2$

$= 4a^2p^2$

$=4a(ap^2)$

$\therefore x^2 = 4ay$.

& $(2ap, ap^2)$ lies on $x^2=4ay$. (1)

(ii) $y = \frac{1}{4a}x^2$.

$y' = \frac{1}{2a}x$.

$\therefore m_T = \frac{1}{2a} \cdot 2ap$.

$= p$.

$\therefore m_N = \frac{-1}{p}$.

$\therefore$ N: $y - ap^2 = \frac{-1}{p}(x-2ap)$.

$py - ap^3 = -x + 2ap$.

$\boxed{x + py = 2ap + ap^3}$ (2)

(iii) N is the point $(0, ap^2)$

& G is the point $(0, 2a+ap^2)$

$\therefore NG = 2a$. (1)

Hence the subnormal NG is independent of p and equal to twice the focal length.

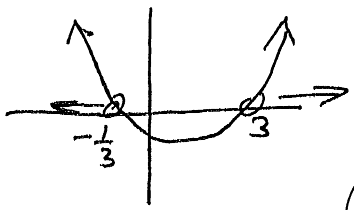
(d) $|2x-1| > |x+2|$

$\therefore (2x-1)^2 > (x+2)^2$ (NB if $a, b > 0$ then $a > b \Rightarrow a^2 > b^2$.)

$4x^2 - 4x + 1 > x^2 + 4x + 4$

$3x^2 - 8x - 3 > 0$

$(3x+1)(x-3) > 0$



$\therefore \boxed{x < -\frac{1}{3},\ x > 3.}$

2

(e) $\cos\theta \sin\theta = \frac{1}{2}$

$\therefore 2\cos\theta \sin\theta = 1$

$\sin 2\theta = 1$

$2\theta = n\pi + (-1)^n \frac{\pi}{2}$

$\boxed{\theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4},\ n \in \mathbb{Z}^*}$

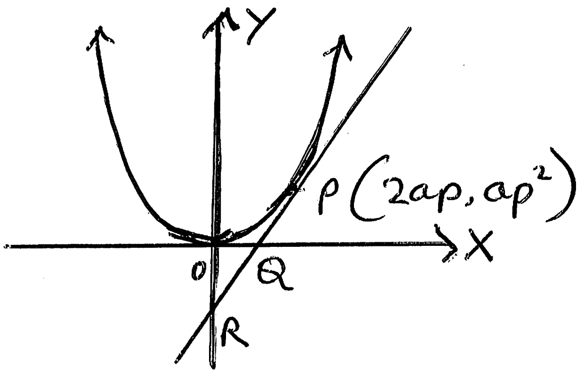
2

QUESTION 4

(a) 1 (i) $5! = 120$ ways (1)

2 (ii) $\dfrac{3 \times 2 \times 3!}{5!} = \dfrac{3}{10}$ (2)

(b) 4



Eq^n tangent at P

$y - px + ap^2 = 0$ (1)

When $x = 0$, $y = -ap^2$

$\Rightarrow R(0, -ap^2)$ ($\frac{1}{2}$)

When $y = 0$, $x = ap$

$\Rightarrow Q(ap, 0)$ ($\frac{1}{2}$)

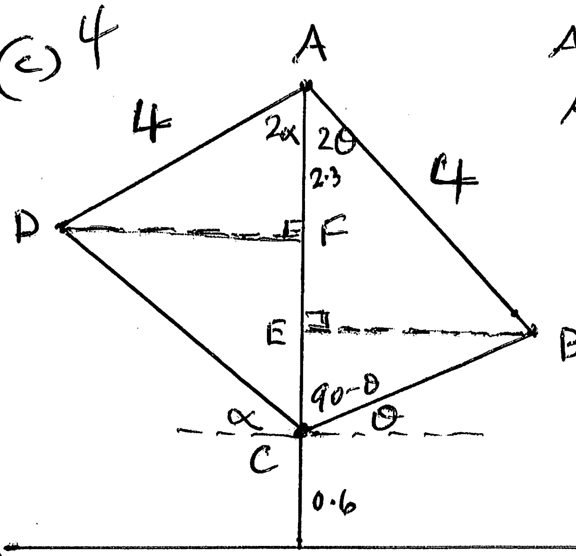
$\therefore M\left(\frac{ap}{2}, \frac{-ap^2}{2}\right)$ ($\frac{1}{2}$)

let $x = \frac{ap}{2}$, $y = -\frac{ap^2}{2}$

$p = \frac{2x}{a}$, $y = -\frac{a}{2}\left(\frac{2x}{a}\right)^2$

locus $y = -\frac{2x^2}{a}$ ($1\frac{1}{2}$)

(c) 4



AC = 4

AE = 2·6

($\frac{1}{2}$)

(i)

In Δ ADF $\cos 2\alpha = \frac{2 \cdot 3}{4}$

$2\alpha = 54°54'1.32'$ ($1\frac{1}{2}$)

In Δ AEB $\cos 2\theta = \frac{2.6}{4}$ (1)

$2\theta = 49°27'30.23''$

$\therefore \hat{DAB} \doteqdot 104°$

(ii)

$(BD)^2 = 4^2 + 4^2 - 2 \cdot 4 \cdot 4 \cos 104°$

$BD \doteqdot 6.3$ m

ie 630 cm (1)

(d)

(i) $\tan(\alpha+\beta+\gamma) = \tan\left[(\alpha+\beta)+\gamma\right]$ (4)

(2)

$$= \frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} + \tan\gamma}{1-\left(\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)\cdot\tan\gamma}$$

(*) or $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha+\tan\beta+\tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$

(ii) $\alpha=\beta=\gamma=\frac{\pi}{12}$ subst in (*)

$\Rightarrow \tan\frac{\pi}{4} = 1 \quad \therefore \quad$ (*) $\Rightarrow$ (2)

$1-\tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma = \tan\alpha+\tan\beta+\tan\gamma - \tan\alpha\tan\beta\tan\gamma$

$1-\tan^2\frac{\pi}{12} - \tan^2\frac{\pi}{12} - \tan^2\frac{\pi}{12} = 3\tan\frac{\pi}{12} - \tan^3\frac{\pi}{12}$

$\Rightarrow \tan^3\frac{\pi}{12} - 3\tan^2\frac{\pi}{12} - 3\tan\frac{\pi}{12} + 1 = 0$

$\therefore \tan\frac{\pi}{12}$ is a root of

$t^3 - 3t^2 - 3t + 1 = 0$